

On some Brownian functionals and their applications to moments in lognormal and Stein stochastic volatility models

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Abstract

The aim of this paper is to present the new results concerning some functionals of Brownian motion with drift and present their applications in financial mathematics. We find a probabilistic representation of the Laplace transform of special functional of geometric Brownian motion using the squared Bessel and radial Ornstein-Uhlenbeck processes. Knowing the transition density functions of the above we obtain computable formulas for certain expectations of the concerned functional. As an example we find the moments of processes representing an asset price in the lognormal volatility and Stein models. We also present links among the geometric Brownian motion, the Markov processes studied by Matsumoto and Yor and the hyperbolic Bessel processes.

Key words: geometric Brownian motion, Ornstein-Uhlenbeck process, Laplace's transform, Bessel process, hyperbolic Bessel process

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1 Introduction

The aim of this paper is to present the new results concerning some functionals of Brownian motion with drift and their applications to financial mathematics. The laws of many different functionals of Brownian motion have been studied in recent years (see, among others, [6], [4], [7], [19], [20]), but some of the obtained results can not be effectively used in application. The distribution of $\int_0^t e^{B_u^{(\mu)}} du$, where $B_t^{(\mu)} = B_t + \mu t$ with a standard Brownian motion B , is an

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example of such situation. This distribution can be characterized by Hartman-Watson distribution, but the oscillating nature of the last causes the difficulties in numerical calculations connected with this functional (see [2]). We study the laws of special functionals of geometric Brownian motion, and find results convenient for numerical applications. We investigate the functionals of geometric Brownian motion $Y_t^{(\mu)} := \exp(B_t + \mu t)$ for $\mu \in R$. In particular we study properties of the functionals $\Gamma_t = \frac{Y_t^{(-1/2)}}{1 + \beta \int_0^t Y_s^{(-1/2)} ds}$ and $1 + \beta A_t^{(\mu)}$ for $\beta > 0$, where $A_t^{(\mu)} := \int_0^t (Y_u^{(\mu)})^2 du$. We deliver the probabilistic representation for Laplace transform of Γ . In our probabilistic representation of Laplace transform of Γ we use the squared Bessel and radial Ornstein-Uhlenbeck processes. Knowing the transition density functions of these processes we obtain computable formulas for certain expectations of the concerned functionals. One of the many advantages of the new result is the fact that they can be effectively used in numerical computations. As an example we compute the moments $\mathbb{E}X_t^\alpha$, for $\alpha > 0$, of the processes X_t representing an asset price in an important stochastic volatility model - in the lognormal volatility model. The necessity of computing moments results from the problems of pricing derivatives (for instance, the broad class of interest rate derivatives necessity a "convexity correction" to the forward rate price; for details see e.g. [5]) as well as from the need of approximations of characteristic functions of random variables with very complicated distributions.

We now give a detailed plan of this paper. In subsection 2.1 we present a method of calculating the moments $\mathbb{E}\Gamma_t^k$ for $k \in \mathbb{Z}$ (Proposition 2.2, Remarks 2.3, Corollary 2.10) and investigate the connection of functional Γ with a hyperbolic Bessel process (Theorem 2.4). The general connections between hyperbolic Bessel processes and functionals of geometric Brownian motion are presented in [12]. In subsection 2.2 we investigate the different properties of $1 + \beta A_t^{(\mu)}$. We find two different probabilistic representations of the Laplace transform of $(1 + \beta A_t^{(\mu)})^{-1}$ (Theorems 2.5 and 2.6), the form of $\mathbb{E} \ln(1 + \beta A_t^{(\mu)})$ and $\mathbb{E}(1 + \beta A_t^{(\mu)})^{-1}$ (Theorem 2.8). It turns out that for an arbitrary strictly positive random variable we can find a representation of the Laplace's transform of $(s + \xi)^{-1}$ for $s \geq 0$, in terms of a squared Bessel process (Theorem 2.11). Moreover, we find some interesting connections between $\mathbb{E}((1 + \beta A_t^{(\mu)})^{-1})$ and the conditional expectation of functionals of geometric Brownian motion with opposite drift. Notice that we establish all results for a fixed t . Section 3 is an illustration of using the previous results in mathematical finance. We assume that the asset price process X satisfies $dX_t = Y_t X_t dW_t$ with Y being a (GBM) (this model is called the lognormal stochastic volatility model or the Hull-White model, see [10]) and Y being an Ornstein-Uhlenbeck process (OU) (the Stein model, see [24]). The distribution of the asset price for the lognormal stochastic volatility model is known but degree of complication and numerical obstacles encourage to look for simpler approximations. Jourdain [13] has given conditions on existence of the moments, if Y is a (GBM), but not mentioned about how to compute it. In this work we find that the moment is equal to the Laplace's transform of the process Γ (Theorem 3.1). We can also express moments of order $\alpha > 1$ in terms of the hyperbolic Bessel process (Theorem 3.4). For the model with random time T_λ being an exponential random variable independent of Brownian motion driving the diffusion Y we find the closed formula of $\mathbb{E}X_{2T_\lambda}^\alpha$.

(Theorem 3.6). In Proposition 3.7 we give a closed formula for moments in the Stein model.

Summing up, we present forms of some interesting functionals of Brownian motion. Moreover, we find interesting links among a GBM, Markov processes arisen during generalization of the so called Pitmann's $2M - X$ theorem (see [17], [18], [19], [20]) and a hyperbolic Bessel process (see [21]). Finally, we compute the moments of the asset price process in the lognormal stochastic volatility and Stein models.

2 Properties of some functionals of geometric Brownian motion

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space with filtration $\mathbb{F} = (\mathcal{F}_t)_{t \in [0, \infty)}$ satisfying the usual conditions. Let the process Y be of the form

$$Y_t = \exp\left(B_t - \frac{t}{2}\right), \quad (1)$$

where B is a Brownian motion. Functionals of Y play a crucial role in many problems of modern stochastic analysis. The studies of the properties of integral $\int_0^t Y_u^2 du$ are motivated by the problem of pricing Asian options. (see [15], [7]). The process $Y_t^{-2} \int_0^t Y_u^2 du$ has been considered by Matsumoto and Yor in several works concerning laws of Brownian motion functionals. Along with $Y_t^{-1} \int_0^t Y_u^2 du$ it plays a central role in a generalization of the Pitmann's $2M - X$ theorem (for details see for instance [17], [18], [20]). Here, we investigate, among others, the properties of functional Γ defined, for $\beta > 0$, by

$$\Gamma_t = \frac{Y_t}{1 + \beta \int_0^t Y_s ds}. \quad (2)$$

It turns out that this process plays a crucial role in the problem of computing the moments of the asset price in the lognormal stochastic volatility model (see Section 3). We also find some new properties of the exponential functional

$$A_t^{(\mu)} := \int_0^t (Y_u^{(\mu)})^2 du, \quad (3)$$

where, for $\mu \in \mathbb{R}$,

$$Y_t^{(\mu)} := \exp\left(B_t + \mu t\right). \quad (4)$$

Therefore, Y defined by (1) is by definition equal to $Y^{(-1/2)}$, so Γ is a functional of $Y^{(-1/2)}$. We also consider the random variable (which is often called a perpetuity in the mathematical finance literature):

$$A_\infty^{(\mu)} := \int_0^\infty (Y_u^{(\mu)})^2 du. \quad (5)$$

We start from investigation of Γ .

2.1 Some properties of Γ

Proposition 2.1. *If Γ is given by (2), then $\Gamma_0 = 1$ and*

$$d\Gamma_t = \Gamma_t dB_t - \beta \Gamma_t^2 dt. \quad (6)$$

Proof. It follows easily from the Itô lemma. \square

Proposition 2.2. *Let Γ be given by (2) and $p_k(t) = \int_0^t \mathbb{E} \Gamma_u^k du$, $k \in \mathbb{Z}$, $t \in [0, T]$. Then the sequence of functionals (p_k) satisfies the following recurrences:*

$$p'_k(t) = 1 + \frac{k(k-1)}{2} p_k(t) - \beta k p_{k+1}(t), \quad (7)$$

and

$$p_1(t) = \frac{1}{\beta} \mathbb{E} \left(\ln(1 + \beta \int_0^t Y_u du) \right). \quad (8)$$

Proof. By Proposition 2.1 and the Itô lemma we have

$$\Gamma_t^k = 1 + k \int_0^t \Gamma_u^k dB_u - k\beta \int_0^t \Gamma_u^{k+1} du + \frac{k(k-1)}{2} \int_0^t \Gamma_u^k du. \quad (9)$$

The local martingale $\int_0^t \Gamma_u^k dB_u$ is a true martingale as

$$\mathbb{E} \int_0^t \Gamma_u^{2k} du \leq \mathbb{E} \int_0^t Y_u^{2k} du < \infty.$$

Taking expectation of both sides of (9) we obtain (7). Further

$$\begin{aligned} p'_1(t) &= \mathbb{E} \left(\frac{Y_t}{1 + \beta \int_0^t Y_u du} \right) = \frac{1}{\beta} \mathbb{E} \frac{\partial}{\partial t} \left(\ln \left(1 + \beta \int_0^t Y_u du \right) \right) \\ &= \frac{1}{\beta} \frac{\partial}{\partial t} \mathbb{E} \left(\ln \left(1 + \beta \int_0^t Y_u du \right) \right), \end{aligned}$$

as $\ln(1 + \beta \int_0^t Y_u du) \leq \ln(1 + \beta \int_0^T Y_u du)$ and $\mathbb{E} \ln(1 + \beta \int_0^T Y_u du) < \infty$, which implies (8). \square

Remark 2.3. Since, by (7),

$$\mathbb{E} \Gamma_t^k = \frac{k(k-1)}{2} p_k(t) - \beta k p_{k+1}(t), \quad (10)$$

Proposition 2.2 allows to compute $\mathbb{E} \Gamma_t^k$ for $k \in \mathbb{Z}$. Taking $k = -1$, we easily obtain from (7) that

$$p'_{-1}(t) = 1 + \beta t + p_{-1}(t), \quad p_{-1}(0) = 0.$$

This solution is given by the formula $p_{-1}(t) = (\beta - 1)e^t + \beta t + 1 + \beta$. Notice that, having p_{-1} we get recursively from (7) the functions p_{-2}, p_{-3}, \dots . Using the function p_1 we can establish p_2, p_3, \dots . So, using (10), we can find all moments $\mathbb{E} \Gamma_t^k$ for $k \in \mathbb{Z}$, provided we know the form of p_1 . Therefore, to finish this computation we need to find the closed form of the function p_1 . The function p_1 is given by (8), so we have to find $\mathbb{E} \left(\ln(1 + \beta \int_0^t Y_u du) \right)$. The form of p_1 is presented in Corollary 2.10.

Now we investigate the connection of Γ with hyperbolic Bessel processes. Let us recall that a diffusion R with the generator given by

$$\mathcal{A} = \frac{1}{2} \frac{d^2}{dx^2} + \left(\alpha + \frac{1}{2}\right) \coth(x) \frac{d}{dx}, \quad (11)$$

for $\alpha \in \mathbb{R}$, is called a hyperbolic Bessel (HB) process with the parameter α (see [21] or [3]). Therefore R satisfies

$$dR_t = dB_t + \left(\alpha + \frac{1}{2}\right) \coth(R_t) dt. \quad (12)$$

We express the Laplace's transform of functional Γ in terms of the Laplace's transform of \cosh of R .

Theorem 2.4. *Let R be a hyperbolic Bessel process with the parameter $\alpha = -1$ and Γ be given by (2). For $\lambda \geq 0$ we have:*

$$\mathbb{E}e^{-\lambda\Gamma_t} = \mathbb{E}e^{-\beta(\cosh(R_t)-1)}, \quad (13)$$

where the initial value of the process R satisfies $\cosh(R_0) = \frac{\lambda}{\beta} + 1$.

Proof. Let $\theta_t = \beta\Gamma_t$. Then, by (6)

$$d\theta_t = \theta_t dZ_t - \theta_t^2 dt \quad (14)$$

and $\theta_0 = \beta$. Moreover, for $x \geq 0$

$$de^{-x\theta_t} = -e^{-x\theta_t}(x\theta_t dZ_t - x\theta_t^2 dt) + \frac{1}{2}e^{-x\theta_t}x^2\theta_t^2 dt.$$

Taking $p(t, x) := \mathbb{E}e^{-x\theta_t}$ we get from the last expression that p satisfies the PDE

$$\frac{\partial p}{\partial t} = \left(x + \frac{1}{2}x^2\right) \frac{\partial^2 p}{\partial x^2}, \quad (15)$$

with $p(0, x) = e^{-x\beta}$. Therefore, the Laplace's transform of θ_t for $\lambda \geq 0$ is a solution of (15). Consider a stochastic differential equation (SDE)

$$dH_t = \sqrt{H_t^2 + 2H_t} dB_t, \quad H_0 = \frac{\lambda}{\beta} \geq 0. \quad (16)$$

Since for any $0 \leq y \leq x$

$$\left| \sqrt{x^2 + 2x} - \sqrt{y^2 + 2y} \right| \leq \sqrt{(x-y)^2 + 2(x-y)} \quad (17)$$

there exists a weak solution to SDE (16) and the trajectory uniqueness holds for (16) (see [14, Theorem 5.5.4] and [22, Theorem 5.40.1]). Thus, by the Feynman-Kac theorem (after changing terminal condition to the initial one in the Cauchy problem (15)) we obtain that the function $u(t, x) := \mathbb{E}_x e^{-\beta H_t}$ is the unique bounded solution of (15) with $p(0, x) = e^{-x\beta}$ (see [14, Theorem 5.7.6]). Let us define the diffusion $S_t := H_t + 1$. It is easy to check that

$$dS_t = \sqrt{S_t^2 - 1} dB_t, \quad H_0 = \frac{\lambda}{\beta} + 1. \quad (18)$$

By the same arguments as before there exists a weak solution to (18) and the trajectory uniqueness holds for (18). Now, we observe that the diffusion $U_t = \cosh(R_t)$, where R is the hyperbolic Bessel process with the parameter -1 , and such that $\cosh(R_0) = a + 1$ is the solution of (18). So, the processes S and U have the same law. Thus,

$$\mathbb{E}e^{-\lambda\Gamma_t} = \mathbb{E}e^{-\frac{\lambda}{\beta}\theta_t} = p(t, \lambda/\beta) = \mathbb{E}e^{-\beta H_t} = e^\beta \mathbb{E}e^{-\beta S_t} = e^\beta \mathbb{E}e^{-\beta \cosh(R_t)}.$$

This ends the proof. \square

The connections between hyperbolic Bessel processes and functionals of geometric Brownian motion are presented in [12].

2.2 Some properties of $(1 + \beta A_t^{(\mu)})$

We start from the computation of the Laplace's transform of $(1 + \beta A_t^{(\mu)})^{-1}$. It is worth to remark that we compute it for a fixed time t . We can find in literature (see for instance [19]) that the problem of computing of expectations for functionals of geometric Brownian motion for a fixed time is in general much difficult than with stochastic one (see also Subsection 3.2.2).

Let us recall that a squared δ -dimensional radial Ornstein-Uhlenbeck process with the parameter $-\lambda$ for $\delta \geq 0, \lambda \in \mathbb{R}$, is the solution of the SDE

$$X_t = x + \int_0^t (\delta - 2\lambda X_s) ds + 2 \int_0^t \sqrt{X_s} dW_s, \quad (19)$$

where W is a standard Brownian motion. For detailed studies of these processes see [9] and [4]. If $\lambda = 0$, then the strong solution of (19) is a squared δ -dimensional Bessel process (see [21]). The number $\delta/2 - 1$ is called the index of the process. In the sequel we will use the notation X^x for the process X starting from x , i.e. $X_0 = x$.

In the next two theorems we find a probabilistic representation of the Laplace transform of $(1 + \beta A_t^{(\mu)})^{-1}$.

Theorem 2.5. *Assume $\beta \in (0, 1]$, $\mu \in \mathbb{R}$ and $t > 0$. Then, for any $\lambda > 0$*

$$\mathbb{E} \exp \left(- \frac{\lambda}{1 + \beta A_t^{(\mu)}} \right) = \mathbb{E} \phi_t(\theta^\lambda(-\ln(\sqrt{\beta}))), \quad (20)$$

where $\theta^\lambda(t)$ is a squared 0-dimensional radial Ornstein-Uhlenbeck process with the parameter -1 such that $\theta(0) = \lambda$ and, for $x > 0$,

$$\phi_t(x) = \psi_t(1, x), \quad (21)$$

and for $x > 0, s \geq 0$

$$\psi_t(s, x) = \mathbb{E} G_t(R^x(s/2)). \quad (22)$$

Here R^x is a squared Bessel process with the index -1 starting from x , and

$$G_t(x) = e^{-t\mu^2/2} \mathbb{E} \exp \left(\mu B_t + \frac{1}{2t} (B_t^2 - \varphi_x^2(B_t)) \right), \quad (23)$$

B is a standard Brownian motion, and

$$\varphi_x(y) = \ln \left(x e^{-y} + \cosh(y) + \sqrt{x^2 e^{-2y} + \sinh^2(y) + 2x e^{-y} \cosh(y)} \right).$$

Proof. Let B be a standard Brownian motion under \mathbb{P} . Let us define the function

$$q(s, x) := \mathbb{E} \exp \left(- \frac{x}{1 + sA_t^{(\mu)}} \right) \quad (24)$$

for $x \geq 0$, $s \in [0, 1]$. Observe that $q(s, x) \leq 1$. It is not difficult to check, using the Lebesgue theorem, that q belongs to the class $C^{1,2}([0, \infty) \times [0, \infty))$. Moreover, it is easy to see that q satisfies the partial differential equation

$$-s \frac{\partial q}{\partial s} = x \left(\frac{\partial q}{\partial x} + \frac{\partial^2 q}{\partial x^2} \right), \quad (25)$$

and $q(s, 0) = 1$. Define

$$\phi_t(x) := q(1, x) = \mathbb{E} \exp \left(- \frac{x}{1 + A_t^{(\mu)}} \right) \quad \text{and} \quad p(s, x) := q(e^{-s}, x).$$

Then $p(s, x)$ satisfies the partial differential equation

$$\frac{\partial p}{\partial s} = x \left(\frac{\partial p}{\partial x} + \frac{\partial^2 p}{\partial x^2} \right), \quad (26)$$

$s \geq 0, x \geq 0$ and $p(0, x) = \phi_t(x)$. Consider a diffusion U with a generator of the form

$$\mathcal{A}_U = x \frac{d^2}{dx^2} + x \frac{d}{dx}. \quad (27)$$

This diffusion satisfies the SDE

$$dU_t = \sqrt{2} \sqrt{U_t} dW_t + U_t dt, \quad (28)$$

where W is a standard Brownian motion. Since $p(s, x) \leq 1$ and $p \in C^{1,2}([0, \infty) \times [0, \infty))$, using the Feynman-Kac theorem (after changing terminal condition to the initial one in the Cauchy problem (26)) we obtain that p admits the stochastic representation $p(s, x) = \mathbb{E} \phi_t(U_s^x)$ (see [14, Theorem 5.7.6]).

Observe that $\theta(s) := U_{2s}$ is a 0-dimensional radial Ornstein-Uhlenbeck process with the parameter -1 , as (28) takes, by the scaling property of Brownian motion, the form

$$d\theta(t) = 2\sqrt{\theta(t)} dB_t + 2\theta(t) dt. \quad (29)$$

Thus

$$q(s, x) = p(-\ln s, x) = \mathbb{E} \phi_t(\theta^x(-(1/2) \ln s)) = \mathbb{E} \phi_t(\theta^x(-\ln \sqrt{s})), \quad (30)$$

$\theta(0) = x$. Hence, taking $x = \lambda, s = \beta$ we see that (30) gives (20):

$$\mathbb{E} \exp \left(- \frac{\lambda}{1 + \beta A_t^{(\mu)}} \right) = q(s, \lambda) = \mathbb{E} \phi_t(\theta^\lambda(-\ln(\sqrt{\beta}))).$$

To finish the proof we have to find the form of ϕ_t . To do this we define the new functions:

$$\psi_t(s, x) := \mathbb{E} \exp \left(- \frac{x}{s + A_t^{(\mu)}} \right), \quad (31)$$

$$G_t(x) := \mathbb{E} \exp \left(- \frac{x}{A_t^{(\mu)}} \right), \quad (32)$$

for $s \geq 0, x \geq 0$. So $\psi_t(1, \lambda) = \phi_t(\lambda)$ and $G_t(x) = \psi_t(0, x)$. Observe that ψ_t satisfies the partial differential equation

$$\frac{\partial \psi_t}{\partial s} = x \frac{\partial^2 \psi_t}{\partial x^2}. \quad (33)$$

Consider a diffusion X with a generator of the form

$$\mathcal{A}_X = x \frac{d^2}{dx^2}.$$

Using again the Feynman-Kac theorem we deduce that ψ admits the stochastic representation $\psi_t(s, x) = \mathbb{E}G_t(X_s^x)$. Now observe that $R(s) := X_{2s}^x$ satisfies

$$dR(t) = 2\sqrt{R(t)}dB_t, \quad (34)$$

so R is a squared Bessel process with the index -1 . Therefore, we obtain (22). It remains to compute the form of the function G_t . Define a new probability measure \mathbb{Q} by

$$\frac{d\mathbb{Q}}{d\mathbb{P}} \Big|_{\mathcal{F}_t} = \exp \left(-\mu B_t - \frac{\mu^2}{2}t \right). \quad (35)$$

Since B is a standard Brownian motion under \mathbb{P} , then $\widehat{B}_t = B_t + \mu t$ is a standard Brownian motion under \mathbb{Q} , by the Girsanov theorem. For $\widehat{A}_t^{(0)} := \int_0^t e^{2\widehat{B}_u} du$ we have $A_t^{(\mu)} = \widehat{A}_t^{(0)}$, so

$$G_t(x) = \mathbb{E}e^{-\frac{x}{A_t^{(\mu)}}} = \mathbb{E}_{\mathbb{Q}} \left(e^{-\frac{x}{\widehat{A}_t^{(0)}}} e^{\mu B_t + \frac{\mu^2}{2}t} \right) = e^{-\mu^2 t/2} \mathbb{E}_{\mathbb{Q}} \exp \left(\mu \widehat{B}_t - \frac{x}{\widehat{A}_t^{(0)}} \right).$$

Now we use the Matsumoto-Yor result [19, Thm. 5.6], which states that

$$\mathbb{E}_{\mathbb{Q}} \left(\exp \left(-\frac{x}{\widehat{A}_t^{(0)}} \right) \Big| \widehat{B}_t = y \right) = \exp \left(-\frac{\varphi_x^2(y) - y^2}{2t} \right),$$

where

$$\begin{aligned} \varphi_x(y) &= \arg \cosh(xe^{-y} + \cosh(y)) \\ &= \ln \left(xe^{-y} + \cosh(y) + \sqrt{x^2 e^{-2y} + \sinh^2(y) + 2xe^{-y} \cosh(y)} \right). \end{aligned}$$

In result we obtain

$$G_t(x) = e^{-t\mu^2/2} \mathbb{E} \exp \left(\mu B_t + \frac{1}{2t} \left(B_t^2 - \varphi_x^2(B_t) \right) \right),$$

where B is a standard Brownian motion under \mathbb{P} . This finishes the proof. \square

Our next theorem provides another probabilistic representation of Laplace transform of $(1 + \beta A_t^{(\mu)})^{-1}$, now for $\beta > 0$.

Theorem 2.6. *Fix $\beta > 0, \mu \in \mathbb{R}$ and $t \geq 0$. Then, for any $\lambda \geq 0$,*

$$\mathbb{E} \exp \left(-\frac{\lambda}{1 + \beta A_t^{(\mu)}} \right) = \mathbb{E}G_t(R^{\lambda/\beta}(1/(2\beta))), \quad (36)$$

where $R^{\lambda/\beta}$ is a squared Bessel process with the index λ/β starting from x and G_t is defined by (23).

Proof. Formula (36) follows from the proof of Theorem 2.5, since for $s \geq 0, x \geq 0$ we have, by (31) and (22),

$$\mathbb{E} \exp \left(- \frac{x}{s + A_t^{(\mu)}} \right) = \psi_t(s, x) = \mathbb{E} G_t(R^x(s/2)) \quad (37)$$

and taking $x = \lambda/\beta$ and $s = 1/\beta$ we obtain (36). \square

Corollary 2.7. *For $\beta \in (0, 1]$ we have*

$$\mathbb{E} \phi_t(\theta^\lambda(-\ln(\sqrt{\beta}))) = \mathbb{E} G_t(R^{\lambda/\beta}(1/(2\beta))), \quad (38)$$

where θ^λ is defined in Theorem 2.5.

Proof. (38) follows from (36) and (20). \square

Using this result we can obtain the expectations of $\ln(1 + \beta A_t^{(\mu)})$ and $(1 + \beta A_t^{(\mu)})^{-1}$.

Theorem 2.8. *Fix $\beta > 0, \mu \in \mathbb{R}$ and $t \geq 0$. Then*

$$\mathbb{E} \ln(1 + \beta A_t^{(\mu)}) = \int_0^\infty \frac{G_t(y)}{y} (1 - e^{-y\beta}) dy, \quad (39)$$

$$\mathbb{E} \left(\frac{1}{1 + \beta A_t^{(\mu)}} \right) = 1 - \beta \int_0^\infty G_t(y) e^{-y\beta} dy, \quad (40)$$

where G_t is given by (23).

Proof. Let $f(\beta) = \mathbb{E} \ln(1 + \beta A_t^{(\mu)})$ for $\beta > 0$. Since

$$\mathbb{E} |\ln(1 + \beta A_t^{(\mu)})| \leq 1 + \beta \mathbb{E} A_t^{(\mu)} < \infty,$$

the function f is well defined, continuous and $f(0) = 0$. Moreover, for $\beta > 0$,

$$f'(\beta) = \mathbb{E} \left(\frac{A_t^{(\mu)}}{1 + \beta A_t^{(\mu)}} \right) = \frac{1}{\beta} \left(1 - \mathbb{E} \left(\frac{1}{1 + \beta A_t^{(\mu)}} \right) \right). \quad (41)$$

By definition of ψ (see (31)) and (22) we know that for $s \geq 0, x \geq 0$ we have

$$\mathbb{E} \exp \left(- \frac{x}{s + A_t^{(\mu)}} \right) = \psi_t(s, x) = \mathbb{E} G_t(R^x(s/2)), \quad (42)$$

where R^x is a Bessel process with the index -1 starting from x . Since, by definition (see (32)), $G_t(0) = 1$ and the transition density functions for the process R^x are known (see [21, Chapter IX, Corollary 1.4]) we can write

$$\mathbb{E} G_t(R^x(s/2)) = e^{-x/s} + \int_0^\infty G_t(y) \frac{1}{s} \sqrt{\frac{x}{y}} e^{-(x+y)/s} I_1(2\sqrt{xy}/s) dy, \quad (43)$$

where I_1 is the modified Bessel function. Let us recall that $(I_1(x)/x)' = I_2(x)/x$ (see [4, Appendix 2]). Hence and by (43) and (42) we obtain, for $x \in [0, 1]$,

$$\begin{aligned} -\mathbb{E}\left(\frac{1}{s + A_t^{(\mu)}} \exp\left(-\frac{x}{s + A_t^{(\mu)}}\right)\right) &= \frac{\partial \psi(s, x)}{\partial x} \\ &= -\frac{1}{s} e^{-x/s} + \frac{\partial}{\partial x} \left(\int_0^\infty G_t(y) \frac{1}{s} \sqrt{\frac{x}{y}} e^{-(x+y)/s} I_1(2\sqrt{xy}/s) dy \right) \\ &= -\frac{1}{s} e^{-x/s} + \frac{2e^{-x/s}}{s^2} \left(1 - \frac{x}{s}\right) \left(\int_0^\infty G_t(y) e^{-y/s} (2\sqrt{xy}/s)^{-1} I_1(2\sqrt{xy}/s) dy \right) \\ &\quad + \frac{2e^{-x/s}}{s^2} \left(\int_0^\infty \sqrt{x} G_t(y) e^{-y/s} (2\sqrt{xy}/s)^{-1} I_2(2\sqrt{xy}/s) \frac{\sqrt{y}}{s} dy \right). \end{aligned}$$

Let x tend to 0. Since we can pass with the limit under the integrals, using the asymptotic behavior of Bessel functions, i.e. $I_1(x)/x \simeq 1/2$ and $I_2(x)/x \simeq x/8$ in neighborhood of 0 (see [4, Appendix 2]) we obtain

$$\mathbb{E}\left(\frac{1}{s + A_t^{(\mu)}}\right) = \frac{1}{s} - \frac{1}{s^2} \int_0^\infty G_t(y) e^{-y/s} dy, \quad (44)$$

or equivalently

$$\mathbb{E}\left(\frac{1}{1 + \frac{1}{s} A_t^{(\mu)}}\right) = 1 - \frac{1}{s} \int_0^\infty G_t(y) e^{-y/s} dy. \quad (45)$$

Putting $s = 1/\beta$ in (45) yields (40). From (41) and (40) we conclude

$$f'(\beta) = \int_0^\infty G_t(y) e^{-y\beta} dy.$$

This finishes proof of the theorem, since $f(0) = 0$. \square

Remark 2.9. Formula (40) gives the closed expression of $\mathbb{E}((1 + \beta A_t^{(\mu)})^{-1})$ for $\beta > 0$. The density of $A_t^{(\mu)}$ is known in literature, but due to complicated nature of Hartman-Watson distribution, it can hardly be used for numerical computations (see for instance [19] and [2]). Since the simple form of function G_t is given explicitly, the formulae (39) and (40) allows to obtain numerically $\mathbb{E} \ln(1 + \beta A_t^{(\mu)})$ and $\mathbb{E}((1 + \beta A_t^{(\mu)})^{-1})$.

Theorem 2.8 allows to find the first function $p_1(\cdot)$ for the recurrence established in Proposition 2.2.

Corollary 2.10. *Let p_1 be given by (8). Then*

$$p_1(t) = \frac{1}{\beta} \int_0^\infty \frac{G_{t/4}(y)}{y} (1 - e^{-4\beta y}) dy, \quad (46)$$

where G is defined by (23).

Proof. Since $Z_u = (1/2)B_{4u}$ is a standard Brownian motion, we infer that

$$\begin{aligned} p_1(4t) &= \mathbb{E} \ln(1 + \beta \int_0^{4t} Y_u du) = \mathbb{E} \ln(1 + 4\beta \int_0^t e^{B_{4u} - 2u} du) \\ &= \mathbb{E} \ln(1 + 4\beta \int_0^t e^{2(Z_u - u)} du) = \mathbb{E} \ln(1 + 4\beta A_t^{(-1)}). \end{aligned}$$

\square

Let us now observe that we can deduce more general fact from the proof of Theorems 2.5 and 2.8. It turns out that for an arbitrary strictly positive random variable we can find a representation of the Laplace's transform of $(s + \xi)^{-1}$ for $s \geq 0$, in terms of a squared Bessel process. It gives also a second simple proof of general version of (22).

Theorem 2.11. *Let ξ be a strictly positive random variable. Then for any $x \geq 0, s \geq 0$*

$$\mathbb{E} \exp \left(-\frac{x}{s + \xi} \right) = \mathbb{E} G(R^x(s/2)), \quad (47)$$

where R^x is a squared Bessel process with the index -1 starting from x and

$$G(x) = \mathbb{E} \exp \left(-\frac{x}{\xi} \right). \quad (48)$$

Moreover for $\beta \geq 0$

$$\mathbb{E} \ln(1 + \beta \xi) = \int_0^\infty \frac{G(y)}{y} (1 - e^{-y\beta}) dy. \quad (49)$$

Proof. Let's take a copy of R^x independent of ξ . Then (48) implies

$$\begin{aligned} \mathbb{E} G(R^x(s/2)) &= \mathbb{E} \exp \left(-\frac{R^x(s/2)}{\xi} \right) = \mathbb{E} \mathbb{E} \left(\exp \left(-\xi^{-1} R^x(s/2) \right) \middle| \xi \right) \\ &= \mathbb{E} \exp \left(-\frac{x \xi^{-1}}{1 + \xi^{-1}s} \right) = \mathbb{E} \exp \left(-\frac{x}{s + \xi} \right), \end{aligned}$$

where we used the form of Laplace transform of squared Bessel process (see [21, Chapter XI, page 441]). The proof of (49) goes in the same way as in Theorem 2.8. \square

Now, we use formula (40) and the results of Matsumoto and Yor to obtain some interesting connections between $\mathbb{E}((1 + \beta A_t^{(\mu)})^{-1})$ and the conditional expectation of functionals of geometric Brownian motion with opposite drift.

Proposition 2.12. *For $\mu > 0$ and $\beta > 0$ we have*

$$\mathbb{E} \left(\frac{1}{1 + 2\beta A_t^{(\mu)}} \right) = 1 - 2\beta \mathbb{E} \left(A_t^{(-\mu)} | A_\infty^{(-\mu)} = 1/(2\beta) \right). \quad (50)$$

Proof. By the result of Matsumoto and Yor [16, Thm. 2.2] the process $\{B_t^{(-\mu)}, t \geq 0\}$ on the set $\{A_\infty^{(-\mu)} = 1/(2\beta)\}$ has the same distribution as the process $\{B_t^{(\mu)} - \log(1 + 2\beta A_t^{(\mu)}), t \geq 0\}$ for $\mu > 0$. From that we obtain

$$\mathbb{E} \left(A_t^{(-\mu)} | A_\infty^{(-\mu)} = 1/(2\beta) \right) = \mathbb{E} \int_0^t \frac{e^{2B_s^{(\mu)}}}{(1 + 2\beta A_s^{(\mu)})^2} ds = \frac{1}{2\beta} \left(1 - \mathbb{E} \left(\frac{1}{1 + 2\beta A_t^{(\mu)}} \right) \right).$$

\square

Proposition 2.13. *Let $\beta > 0$ and $\mu \in \mathbb{R}$. Then*

$$\mathbb{E} \left(A_t^{(-\mu)} | A_\infty^{(-\mu)} = 1/(2\beta) \right) = \frac{1}{2} \int_0^\infty G_t(y) e^{-y\beta} dy. \quad (51)$$

Proof. It follows from Proposition 2.12 and (40). \square

Proposition 2.14. For $\beta > 0, \mu > 0$ we have

$$\mathbb{E}\left(\frac{e^{2\mu B_t^{(-\mu)}}}{1 + 2\beta A_t^{(-\mu)}}\right) = 1 - 2\beta \mathbb{E}\left(A_t^{(-\mu)} | A_\infty^{(-\mu)} = 1/(2\beta)\right). \quad (52)$$

Proof. Fix $\mu > 0$. Define the new probability measure \mathbb{Q} by

$$\frac{d\mathbb{Q}}{d\mathbb{P}} \Big|_{\mathcal{F}_t} = e^{-2\mu B_t - 2\mu^2 t}. \quad (53)$$

The process $V_t = B_t + 2\mu t$ is a standard Brownian motion under \mathbb{Q} , so

$$\mathbb{E}\left(\frac{1}{1 + 2\beta A_t^{(\mu)}}\right) = \mathbb{E}_{\mathbb{Q}}\left(\frac{e^{2\mu(V_t - \mu t)}}{1 + 2\beta \int_0^t e^{2(V_u - \mu u)} du}\right) = \mathbb{E}\left(\frac{e^{2\mu B_t^{(-\mu)}}}{1 + 2\beta A_t^{(-\mu)}}\right),$$

Now the thesis follows from Theorem 2.12. \square

Proposition 2.15. For $t \geq 0$ we have

$$p_1(t) = t - 4\beta \int_0^t \mathbb{E}\left(A_{s/4}^{(-1)} | A_\infty^{(-1)} = 1/(4\beta)\right) ds. \quad (54)$$

Proof. We have $p_1'(t) = \mathbb{E}\Gamma_t$, $p_1(0) = 0$, and

$$\Gamma_{4t} = \frac{e^{2\overline{B}_t^{(-1)}}}{1 + 4\beta \overline{A}_t^{(-1)}}, \quad (55)$$

where $\overline{B}_t = B_{4t}/2$ is a standard Brownian motion, and $\overline{A}^{(-1)}$ is defined by (3) with \overline{B} instead of B . Since

$$\mathbb{E}\left(A_{t/4}^{(-1)} | A_\infty^{(-1)} = 1/(4\beta)\right) = \mathbb{E}\left(\overline{A}_{t/4}^{(-1)} | \overline{A}_\infty^{(-1)} = 1/(4\beta)\right),$$

the thesis follows from (52) with $\mu = 1$. \square

Remark 2.16. Notice, that we establish all the results for fixed t . In several papers (for instance [15], [19]) the integral functionals of a geometric Brownian motion with random time given by random variable independent of Brownian motion and with exponential distribution were investigated. In particular,

$$\mathbb{E} \ln \left(1 + \beta \int_0^{T_\lambda} Y_u^2 du\right) = \mathbb{E} \ln \left(1 + \beta \frac{\zeta_{1,a}}{\gamma_b}\right), \quad (56)$$

because $\int_0^{T_\lambda} Y_u^2 du \stackrel{d}{=} \frac{\zeta_{1,a}}{\gamma_b}$, where $\zeta_{1,a}$ is a random variable with beta distribution with the parameters 1 and $a = \frac{\sqrt{2\lambda+1/4}-1/2}{2}$, γ_b is a random variable with gamma distribution with the parameter $b = \frac{\sqrt{2\lambda+1/4}+1/2}{2}$, $\zeta_{1,a}$ and γ_b are independent (see [19]). Later, we also explore idea of using random time. In subsection 3.2.2 we show how to compute the moments in a lognormal stochastic volatility model with random time being exponentially distributed and independent of Brownian motion driving the model.

3 Moments of the asset price in the lognormal stochastic volatility and Stein models

3.1 Model of market

We consider a market defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with filtration $\mathbb{F} = (\mathcal{F}_t)_{t \in [0, T]}$, $T < \infty$, satisfying the usual conditions and $\mathcal{F} = \mathcal{F}_T$. Without loss of generality we assume the savings account to be constant and identically equal to one. Moreover, we assume that the price X_t of the underlying asset at time t has a stochastic volatility Y_t being a geometric Brownian motion or an Ornstein-Uhlenbeck process, so the dynamics of the process X is given by

$$dX_t = Y_t X_t dW_t, \quad (57)$$

where $X_0 = 1$. In case of Y being a GBM the dynamics of the vector (X, Y) is given by (57) and

$$dY_t = Y_t dZ_t, \quad Y_0 = 1, \quad (58)$$

and in case of Y being an OU the dynamics of the vector (X, Y) is given by (57) and

$$dY_t = -\lambda Y_t dt + dZ_t, \quad Y_0 = 1 \quad (59)$$

for $\lambda > 0$. The processes W, Z are correlated Brownian motions, $d\langle W, Z \rangle_t = \rho dt$ with $\rho \in [-1, 1]$. In the both cases the process X has the form

$$X_t = e^{\int_0^t Y_u dW_u - \int_0^t Y_u^2 du/2}, \quad (60)$$

and this is a unique strong solution of SDE (57) on $[0, T]$. The existence and uniqueness follow directly from the assumptions on Y and the well known properties of stochastic exponent (see, e.g., Revuz and Yor [21]). Since the process X is a local martingale, there is no arbitrage on the market so defined. Notice that we can represent W as

$$W_t = \rho Z_t + \sqrt{1 - \rho^2} V_t, \quad (61)$$

where (V, Z) is a standard two-dimensional Wiener process. Using (60) and (61) we can express the moment of order α of X as

$$\begin{aligned} \mathbb{E} X_t^\alpha &= \mathbb{E} e^{\alpha \rho \int_0^t Y_u dZ_u + \alpha \sqrt{1 - \rho^2} \int_0^t Y_u dV_u - \frac{\alpha^2}{2} \int_0^t Y_u^2 du} \\ &= \mathbb{E} e^{\alpha \rho \int_0^t Y_u dZ_u + \frac{\alpha^2(1 - \rho^2) - \alpha}{2} \int_0^t Y_u^2 du}. \end{aligned} \quad (62)$$

3.2 Moments of the asset price in the lognormal stochastic volatility model

3.2.1 Moments of order $\alpha > 0$

In this subsection we will calculate moments of order $\alpha > 0$ in the lognormal stochastic volatility model, so for Y of the form (58). Jourdain [13] gave a

sufficient condition on $\alpha > 1$ for existing of moments of X . Namely, Jourdain proved that for $\alpha \in (1, (1 - \rho^2)^{-1})$ and $\rho \neq 0$ the moments EX^α exist, but he didn't find the value of these moments. We calculate the value of moments for $\alpha > 0$, $\alpha(1 - \rho^2) < 1$ and $\rho \in [-1, 1]$. Sin [23] established that the process X is a true martingale if and only if $\rho \leq 0$. First, we prove that the moment of order α of the strong solution of (57) is equal to the Laplace's transform of the process Γ .

Theorem 3.1. *Let $t \in [0, T]$, $\alpha > 0$, $\alpha(1 - \rho^2) < 1$ and Γ be given by (2). If X is given by (57), then*

$$\mathbb{E}X_t^\alpha = e^{-(\beta + \rho\alpha)} \mathbb{E} \exp \left((\beta + \rho\alpha) \Gamma_t \right), \quad (63)$$

where

$$\beta = \sqrt{\alpha - \alpha^2(1 - \rho^2)}. \quad (64)$$

Proof. Define a measure \mathbb{Q} by

$$\frac{d\mathbb{Q}}{d\mathbb{P}} \Big|_{\mathcal{F}_T} = e^{-\beta \int_0^T Y_u dZ_u - \frac{\beta^2}{2} \int_0^T Y_u^2 du}, \quad (65)$$

where β is given by (64). The measure \mathbb{Q} is a probability measure since, by (58),

$$e^{-\beta \int_0^T Y_u dZ_u - \frac{\beta^2}{2} \int_0^T Y_u^2 du} = e^{-\beta(Y_T - 1) - \frac{\beta^2}{2} \int_0^T Y_u^2 du} \leq e^\beta.$$

Using (62) and the definition of \mathbb{Q} we infer

$$\mathbb{E}X_t^\alpha = \mathbb{E} e^{\alpha \rho \int_0^t Y_u dZ_u + \frac{\alpha^2(1 - \rho^2) - \alpha}{2} \int_0^t Y_u^2 du} = \mathbb{E}_{\mathbb{Q}} e^{(\rho\alpha + \beta)(Y_t - 1)}. \quad (66)$$

By the Girsanov theorem, $B_t = Z_t + \int_0^t \beta Y_s ds$ is a standard Brownian motion under \mathbb{Q} and

$$dZ_t = dB_t - \beta Y_t dt, \quad Z_0 = 0. \quad (67)$$

We know, by the result of Alili, Matsumoto and Shiraishi [1, Lemma 3.1], that the unique strong solution of (67) is given by

$$Z_t = \frac{t}{2} + \ln \left(\frac{U_t}{1 + \beta \int_0^t U_s ds} \right),$$

where

$$U_t = e^{B_t - \frac{t}{2}}.$$

Therefore

$$Y_t = \exp \left(Z_t - \frac{t}{2} \right) = \frac{U_t}{1 + \beta \int_0^t U_s ds}.$$

The law of the process Y under \mathbb{Q} is equal to the law of the process Γ under \mathbb{P} , since the law of the process Y under \mathbb{P} is equal to the law of the process U under \mathbb{Q} . Hence and by (66) we obtain (63). This ends the proof. \square

Remark 3.2. a) From Theorem 3.1 we immediately see that all moments exist provided $\rho^2 = 1$.

b) The condition $\alpha(1 - \rho^2) < 1$ is not a necessary condition for existence of moments since in case of $\rho = 0$ the process X is a martingale, so EX_t exists. Although, for $\rho = 0$, $EX_t^\alpha = \infty$ for $\alpha > 1$ ([13]).

Remark 3.3. In Theorem 3.1 we prove that the computation of EX_t^α for $\alpha > 0$ such that $\alpha(1 - \rho^2) < 1$ is equivalent to the computation of the Laplace's transform of Γ_t at point $\lambda = \beta + \rho\alpha$. In turn, the recurrence from Proposition 2.2 allows to find $\mathbb{E}\Gamma_t^k$, so we can find an approximation of the Laplace's transform of Γ in the neighborhood of zero by its moments, namely $\mathbb{E}e^{\lambda\Gamma_t} \approx \sum_{i=0}^N \lambda^i \frac{\mathbb{E}\Gamma_t^i}{i!}$ for sufficiently large N . In this way we obtain an approximate value of EX_t^α .

We can also express moments of order $\alpha > 1$ in terms of the hyperbolic Bessel process with the parameter -1 .

Theorem 3.4. Assume that $\alpha > 1$, $\alpha(1 - \rho^2) < 1$. Let X be given by (57) with Y given by (1) and R be a hyperbolic Bessel process with the parameter -1 . Then

$$\mathbb{E}X_t^\alpha = e^{-\rho\alpha} \mathbb{E}e^{-\beta \cosh(R_t)}, \quad (68)$$

where $\cosh(R_0) = -\rho\alpha/\beta$, $\beta = \sqrt{\alpha - \alpha^2(1 - \rho^2)}$.

Proof. We use Theorem 3.1 and Theorem 2.4 with $\lambda = -(\beta + \rho\alpha)$, and $\lambda > 0$ provided $\alpha > 1$. \square

3.2.2 Moments with independent random time

In this subsection we find the closed formulae for the moments in a lognormal stochastic volatility, when the time is an exponential random variable independent of Brownian motion driving the diffusion Y . The idea of considering such a time is not new and can be find in many studies of Asian options (see for instance [15], [19]).

Proposition 3.5. Let T_λ be a random variable with exponential distribution with the parameter $\lambda > 0$. Assume that T_λ is independent of a standard Brownian motion B . Let $Z_t = 2B_{\frac{t}{4}}$, $Y_t = e^{-\frac{t}{2} + Z_t}$ and $U_t = e^{B_t - t}$. Then

$$\mathbb{E}\left(\ln\left(1 + \beta \int_0^{4T_\lambda} Y_u du\right)\right) = \frac{4\beta}{\lambda} - 4\beta^2 \int_0^\infty \mathbb{E}\left(\int_0^{T_\lambda} U_s^2 ds - K/4\right)^+ (1 + \beta K)^{-2} dK, \quad (69)$$

$$\mathbb{E}\left(\int_0^{T_\lambda} U_s^2 ds - K/4\right)^+ = \frac{1}{\lambda \Gamma\left(\frac{\sqrt{2\lambda+1}-1}{2}\right)} \int_0^{2/K} e^{-u} u^{\frac{\sqrt{2\lambda+1}-3}{2}} (1 - Ku/2)^{\frac{\sqrt{2\lambda+1}+1}{2}} du. \quad (70)$$

Proof. It is obvious that $Y_{4t} = e^{-2t+Z_{4t}} = e^{-2t+2B_t} = U_t^2$. Using the Taylor theorem with integral remainder to the function $f(x) = \ln(1 + \beta x)$ gives

$$\ln(1 + \beta x) = \beta x - \beta^2 \int_0^\infty (x - K)^+ (1 + \beta K)^{-2} dK. \quad (71)$$

Hence replacing x by $\int_0^{4T_\lambda} Y_u du$ and taking expectation we get

$$\begin{aligned} \mathbb{E}\left(\ln(1 + \beta \int_0^{4T_\lambda} Y_u du)\right) &= \beta \mathbb{E}\left(\int_0^{4T_\lambda} Y_u du\right) \\ &- \beta^2 \int_0^\infty \mathbb{E}\left(\int_0^{4T_\lambda} Y_u du - K\right)^+ (1 + \beta K)^{-2} dK \\ &= 4\beta \mathbb{E}\left(\int_0^{T_\lambda} U_s^2 ds\right) - 4\beta^2 \int_0^\infty \mathbb{E}\left(\int_0^{T_\lambda} U_s^2 ds - K/4\right)^+ (1 + \beta K)^{-2} dK. \end{aligned} \quad (72)$$

Let $A_t = \int_0^t U_s^2 ds$. The Mansuy and Yor theorem [15, Thm. 6.1] gives (70) and $\mathbb{E}A_{T_\lambda} = 1/\lambda$. This and (72) completes the proof. \square

In the next theorem we establish the explicit formula for moments of $X_{T_{2\lambda}}$.

Theorem 3.6. *Let $\alpha > 0$, $\alpha(1 - \rho^2) < 1$ and T_λ be a random variable with exponential distribution with the parameter $\lambda > 0$. Assume that T_λ is independent of Brownian motions V and Z driving the process X . Then*

$$\begin{aligned} \mathbb{E}X_{2T_\lambda}^\alpha &= \frac{1}{\lambda} e^{-(\alpha\rho+\beta)} \frac{\Gamma((1 + \sqrt{4\lambda+1})/2)}{\Gamma(1 + \sqrt{4\lambda+1})} \times \\ &\left(\phi_1(1/2\beta) \int_0^{1/2\beta} e^{\frac{\alpha\rho-\beta}{2\beta}\frac{1}{y}} \phi_2(y) dy + \phi_2(1/2\beta) \int_{1/2\beta}^\infty e^{\frac{\alpha\rho-\beta}{2\beta}\frac{1}{y}} \phi_1(y) dy \right), \end{aligned} \quad (73)$$

where $\beta = \sqrt{\alpha - \alpha^2(1 - \rho^2)}$,

$$\begin{aligned} \phi_1(x) &= x^{-(1+\sqrt{1+4\lambda})/2} \Phi\left((1 + \sqrt{1+4\lambda})/2, 1 + \sqrt{1+4\lambda}, x^{-1}\right), \\ \phi_2(x) &= x^{-(1+\sqrt{1+4\lambda})/2} \Psi\left((1 + \sqrt{1+4\lambda})/2, 1 + \sqrt{1+4\lambda}, x^{-1}\right), \end{aligned}$$

and Φ, Ψ denote the confluent hypergeometric functions of the first and second kind, respectively

$$\begin{aligned} \Phi(\alpha, \gamma, z) &= \sum_{k=0}^\infty \frac{(\alpha)_k}{(\gamma)_k} \frac{z^k}{k!}, \\ \Psi(\alpha, \gamma, z) &= \frac{\Gamma(1-\gamma)}{\Gamma(1+\alpha-\gamma)} \Phi(\alpha, \gamma, z) + \frac{\Gamma(\gamma-1)}{\Gamma(\alpha)} z^{1-\gamma} \Phi(1+\alpha-\gamma, 2-\gamma, z), \end{aligned}$$

where $(\alpha)_0 = 1$ and

$$(\alpha)_k = \frac{\Gamma(\alpha+k)}{\Gamma(\alpha)} = \alpha(\alpha+1)\dots(\alpha+k-1),$$

for $k = 1, 2, \dots$

Proof. If $B_t = \frac{1}{2}Z_{4t}$, $S_t = e^{B_t-t}$, then $Y_{4t} = S_t^2$. So

$$dS_t = S_t(dB_t - \frac{1}{2}dt), \quad S_0 = 1,$$

and

$$\Gamma_{4t} = \frac{Y_{4t}}{1 + \beta \int_0^{4t} Y_s ds} = \frac{S_t^2}{1 + 4\beta \int_0^t S_u^2 du}. \quad (74)$$

Define a new process

$$\theta_t = \frac{1}{4\beta} \frac{1 + 4\beta \int_0^t S_u^2 du}{S_t^2}. \quad (75)$$

From the Itô lemma

$$d\theta_t = -2\theta_t dB_t + (4\theta_t + 1)dt, \quad \theta_0 = \frac{1}{4\beta}.$$

Now we observe that the diffusion θ has the generator

$$\mathcal{A}_\theta = 2x^2 \frac{d^2}{dx^2} + (4x + 1) \frac{d}{dx}, \quad (76)$$

which is identical with the generator of the process

$$\chi_t = \exp(2B_t + 2t) \left(\frac{1}{4\beta} + \int_0^t \exp(-2B_u - 2u) du \right)$$

since from the Itô lemma

$$d\chi_t = 2\chi_t dB_t + (4\chi_t + 1)dt.$$

Hence and from the fact that $\chi_0 = \theta_0$ we deduce that processes θ and χ have the same distribution. Let us take another Brownian motion $B_t^* = \sqrt{2}B_{\frac{t}{2}}$ and define the process

$$\eta_t = \exp(\sqrt{2}B_t^* + t) \left(\frac{1}{2\beta} + \int_0^t \exp(-\sqrt{2}B_u^* - u) du \right). \quad (77)$$

Using the fact that θ_t and χ_t have the same distribution, we obtain $2\theta_{\frac{t}{2}} =^d \eta_t$. Moreover, we know that η_t is a Markov process with the resolvent

$$U_\lambda f(x) = \frac{\Gamma((1 + \sqrt{4\lambda + 1})/2)}{\Gamma(1 + \sqrt{4\lambda + 1})} \left(\phi_1(x) \int_0^x e^{-\frac{1}{y}} \phi_2(y) f(y) dy + \phi_2(x) \int_x^\infty e^{-\frac{1}{y}} \phi_1(y) f(y) dy \right). \quad (78)$$

(for details see [7, Theorem 3.1]), so we conclude by Theorem 3.1, (74), (75) and definition of η that

$$\mathbb{E}X_{2T_\lambda}^\alpha = e^{-(\alpha\rho+\beta)} \mathbb{E} \exp \left\{ \frac{\alpha\rho+\beta}{2\beta} \frac{1}{\eta_{T_\lambda}} \right\} = \frac{1}{\lambda} e^{-(\alpha\rho+\beta)} U_\lambda f \left(\frac{1}{2\beta} \right),$$

with $f(x) = \exp \left\{ \frac{\alpha\rho+\beta}{2\beta} \frac{1}{x} \right\}$. □

3.3 Moments of the asset price in the Stein model

In this section we consider the Stein model, i.e. the model described by (57) with Y being an Ornstein-Uhlenbeck process, so Y is given by (59):

$$dY_t = -\lambda Y_t dt + dZ_t, \quad Y_0 = 1, \quad \lambda > 0.$$

For t in neighborhood of zero we find an exact value of $\mathbb{E}X_t^\alpha$. Let b be the unique solution of equation

$$b(1 - e^{-2b}) = 2. \quad (79)$$

Proposition 3.7. Let $\alpha > 0$ and ρ be such that $\alpha(1 - \rho^2) < 1$, $\rho < \lambda/\alpha$ and

$$\gamma^2 = \lambda^2 - \alpha^2(1 - \rho^2) + \alpha - 2\lambda\rho\alpha > 0. \quad (80)$$

If $t \in [0, \frac{b}{\lambda})$, then

$$\mathbb{E}X_t^\alpha = e^{(1+t)\beta} \left(\cosh(\gamma t) + \frac{2\beta}{\gamma} \sinh(\gamma t) \right)^{-\frac{1}{2}}, \quad (81)$$

where b is given by (79) and

$$\beta = \frac{1}{2}(\lambda - \rho\alpha).$$

Proof. Define the new measure

$$\frac{d\mathbb{Q}}{d\mathbb{P}} \Big|_{\mathcal{F}_t} = e^{\lambda \int_0^t Y_u dZ_u - \frac{\lambda^2}{2} \int_0^t Y_u^2 du}. \quad (82)$$

Clearly, Y_t is the Gaussian random variable with the mean $e^{-\lambda t}$ and variance $\frac{1}{2\lambda}(1 - e^{-2\lambda t})$. Moreover $\mathbb{E}e^{\frac{\lambda^2}{2} Y_u^2} du < \infty$ for $u < t$, since $\frac{\lambda^2}{2}(1 - e^{-2\lambda t}) < 1$ by assumption on t . Thus, by Jensen inequality,

$$\begin{aligned} \mathbb{E}e^{\frac{\lambda^2}{2} \int_0^t Y_u^2 du} &\leq \mathbb{E} \left(\frac{1}{t} \int_0^t e^{\frac{\lambda^2}{2} Y_u^2} du \right) = \frac{1}{t} \int_0^t \mathbb{E}e^{\frac{\lambda^2}{2} Y_u^2} du \\ &= \frac{1}{t} \int_0^t \frac{e^{e^{-2\lambda u} u \lambda^2 / (2 - u\lambda(1 - \exp(-2\lambda u)))}}{\sqrt{1 - \frac{u\lambda}{2}(1 - e^{-2\lambda u})}} du \\ &< \frac{1}{t} \int_0^t \frac{e^{u\lambda^2/2}}{\sqrt{1 - \frac{u\lambda}{2}(1 - e^{-2\lambda u})}} du < \infty. \end{aligned}$$

So, \mathbb{Q} is a probability measure, by the Novikov criterion. Observe, by the Girsanov theorem, that the process Y is a Brownian motion under \mathbb{Q} starting from 1. Formula (59) implies

$$\int_0^t Y_u dZ_u = \frac{1}{2}(Y_t^2 - (t+1)) + \lambda \int_0^t Y_u^2 du,$$

so by (62) we obtain

$$\begin{aligned} \mathbb{E}X_t^\alpha &= e^{-(t+1)\frac{\alpha\rho-\lambda}{2}} \mathbb{E}_{\mathbb{Q}} e^{\frac{\alpha\rho-\lambda}{2} Y_t^2 + \frac{2\lambda\alpha\rho-\lambda^2+\alpha^2(1-\rho^2)-\alpha}{2} \int_0^t Y_u^2 du} \\ &= e^{(1+t)\beta} \mathbb{E}e^{-\beta B_t^2 - \frac{\gamma^2}{2} \int_0^t B_u^2 du} \\ &= e^{(1+t)\beta} \left(\cosh(\gamma t) + \frac{2\beta}{\gamma} \sinh(\gamma t) \right)^{-\frac{1}{2}} \exp \left(\frac{\beta}{2} - \frac{(\gamma/2 + \beta)e^{\gamma t}}{\cosh(\gamma t) + \frac{2\beta}{\gamma} \sinh(\gamma t)} \right), \end{aligned}$$

where in the last equation we use the form of the Laplace's transform of $(B_t^2, \int_0^t B_u^2 du)$, where B is a Brownian motion starting from 1 (see e.g. [4, formula 1.9.7 page 168]). \square

Remark 3.8. If $\alpha \in (0, 1)$, then (80) holds. If $\alpha \geq 1$, $\alpha(1 - \rho^2) < 1 - 2\lambda\rho$ and $\rho \leq 0$ then (80) holds.

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